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The Vector Potential and Motion of Charged Particles in Axisymmetric Magnetic Fields

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Abstract. From the conventional expansion of a scalar magnetic potential (such as the earth's), an expansion of the vector potential is obtained. This expansion is used for analyzing the motion of charged particles in axisymmetric magnetic fields, with special attention to such fields that do not deviate far from a dipole. The results are compared with those of Quenby and Webber. Finally, the relation between Störmer's first integral and the third adiabatic invariant is traced.

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The vector potential. A curl-free magnetic field, such as that of the earth, is generally expressed by means of a scalar potential V

$$\mathbf{B} = -\text{grad } V \tag{1}$$

Since V is harmonic, it is conveniently expanded in spherical harmonics

$$V(r, \vartheta, \varphi) = \frac{1}{R} \sum_{n,m} \left\{ a_{nm} \left(\frac{R}{r}\right)^{n+1} + b_{nm} \left(\frac{r}{R}\right)^{n} \right\} Y_n^m(\vartheta, \varphi) \tag{2}$$

$$Y_n^m(\vartheta, \varphi) = P_n^m(\vartheta) \exp im\varphi$$

where R is some constant length, e.g. the earth's radius. Occasionally, however, it is useful to express $\mathbf{B}$ in terms of a vector potential $\mathbf{A}$

$$\mathbf{B} = \text{curl } \mathbf{A} \tag{3}$$

If the scalar potential is given as in (2), $\mathbf{A}$ can be found in the following way. First, to reduce the arbitrariness in the choice of $\mathbf{A}$ the Coulomb gage condition is added

$$\text{div } \mathbf{A} = 0 \tag{4}$$

$\mathbf{A}$ is then defined within the gradient of an arbitrary harmonic function and satisfies

$$\nabla^2 \mathbf{A} = 0 \tag{5}$$

Now it can be shown [*Backus*, 1958] that any solenoidal vector $\mathbf{A}$ can be expressed by means of two scalars, Ψ_1 and Ψ_2, in the form

$$\mathbf{A} = \text{curl } \Psi_1 \mathbf{r} + \text{curl curl } \Psi_2 \mathbf{r} \tag{6}$$

and the following identity holds:

$$\text{curl curl } \Psi \mathbf{r} \equiv \text{grad } (\partial/\partial r)(\Psi r) - \mathbf{r} \nabla^2 \Psi \tag{7}$$

In particular, if (5) is also satisfied, Ψ_1 and Ψ_2 can both be chosen to be harmonic [*Smythe*, 1950, section 7.04]. If Ψ is a harmonic function, $\partial(\Psi r)/\partial r$ is one too, and it is evident from (7) that Ψ_2 then adds to $\mathbf{A}$ only the gradient of a harmonic function and contributes nothing to $\mathbf{B}$. Using the remaining freedom in choice of $\mathbf{A}$, Ψ_2 can be set equal to zero, giving

$$\mathbf{A} = \text{curl } \Psi_1 \mathbf{r} \tag{8}$$

and by (7)

$$\mathbf{B} = \text{grad } (\partial/\partial r)(\Psi_1 r) \tag{9}$$

The last equation can be identified with (1). The vector potential is then given by (8), with

$$\Psi_1 = \frac{1}{R} \sum_{nm} \left\{ \frac{a_{nm}}{n} \left(\frac{R}{r}\right)^{n+1} - \frac{b_{nm}}{n+1} \left(\frac{r}{R}\right)^{n} \right\} Y_n^m(\vartheta, \varphi) \tag{10}$$

Axial symmetry. From now on, only the case in which the field is axially symmetric, i.e. does not depend on φ, will be considered. For the time, however, $\mathbf{B}$ will not be restricted to be curl free. Then

$$B_r = \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} (A_\varphi r \sin \vartheta) \tag{11a}$$

$$B_\vartheta = -\frac{1}{r \sin \vartheta} \frac{\partial}{\partial r} (A_\varphi r \sin \vartheta) \tag{11b}$$

If, in addition,

$$B_\varphi = 0 \tag{11c}$$

the index φ will be dropped from A_φ, for the vector potential

$$\mathbf{A} = \mathbf{i}_\varphi A \tag{12}$$

then completely describes **B** as well as satisfying (4). The equation of a line of force in any meridional plane is then

$$dr/B_r = r\, d\vartheta/B_\vartheta$$

Substituting (11), this is integrated at once to give

$$Ar \sin\vartheta = \alpha = \text{const} \tag{13}$$

The last equation has also been derived by *Smythe* [1950, section 7.08] and, from a somewhat different approach, by *Ray* [1963, bottom of p. 9]. If the field is also curl free, by (8)

$$A = -\partial\Psi_1/\partial\vartheta$$

When we drop the index m in (10) and use Legendre polynomials $P_n(\vartheta)$, this gives

$$A = -\frac{1}{R}\sum_n \left\{\frac{a_n}{n}\left(\frac{R}{r}\right)^{n+1} - \frac{b_n}{n+1}\left(\frac{r}{R}\right)^n\right\}\frac{dP_n}{d\vartheta} \tag{14}$$

which upon substitution in (13) gives the relation between r and ϑ on a line of force.

Motion of a charged particle. Consider a particle of rest mass m_0, charge q, and velocity $\mathbf{v}$ moving in an axisymmetric field. Its Lagrangian will be (MKS)

$$L = -m_0c^2(1 - v^2/c^2)^{1/2} + q(\mathbf{v}\cdot\mathbf{A}) \tag{15}$$

Because of symmetry, φ is a cyclic coordinate. Denoting by m the relativistic mass, we obtain the following first integral:

$$p_\varphi = \frac{\partial L}{\partial\dot\varphi} = mr^2 \sin^2\vartheta\dot\varphi + qA_\varphi r \sin\vartheta = \text{const} \tag{16}$$

Since energy is conserved, it is useful to divide (16) by $P = mv$ and to denote the new constant by γ. If ω is the angle between $\mathbf{i}_\varphi$ and $\mathbf{v}$, then

$$\cos\omega = \dot\varphi r \sin\vartheta/v$$

so that (16) becomes

$$\cos\omega = \frac{\gamma}{r\sin\vartheta} - \frac{qA_\varphi}{P} \tag{17}$$

This equation has been derived by *Störmer* [1955; it appears in slightly different form in part 2, equation 49.5] and was used by him, by *Treiman* [1953], by *Ray* [1956], and by *Kellogg and Winckler* [1961] in calculating effects of a ring current around the earth, and also by *Lüst and Schlüter* [1957] who derived it directly from the equation of motion.

Now let (11c) be assumed, so that A_φ becomes A. Then (17) gives

$$(P/q)r \sin\vartheta \cos\omega = (\gamma P/q) - Ar\sin\vartheta$$

If the particle's energy is low enough for the guiding-center approximation to hold, $\cos\omega$ will oscillate rapidly around zero, and the particle's orbit in the (r, ϑ) plane will alternate between the two sides of the line of force:

$$Ar\sin\vartheta = \gamma P/q \tag{18}$$

This can be regarded as the particle's guiding line of force (for a similar approach, see *Ray* [1963]). In (17), $|\cos\omega|$ is always less than unity while, in a near-dipole field (qA_φ/P) can be made as large as we want by going to low enough momentums. Thus, at low momentums the left-hand side of (17) must be the difference between two much larger terms, and the particle does not stray far from the line of force of (18).

Treiman [1953] also derived a method of calculating cutoff momentums (in the cosmic-ray sense, i.e. a criterion for finding when orbits are completely trapped by the field) applicable to fields which do not deviate far from a dipole field. When this is used, the following results are obtained. Assuming no external sources ($b_n = 0$), denoting the dipole moment by M_1, letting $M = M_1\mu_0/4\pi$, and defining the Störmer unit of length

$$R_0 = (qM/P)^{1/2} \tag{19}$$

we find that for a given P (and consequent R_0) only trapped orbits exist when

$$r < R_1 \cong R_0 - \frac{q}{2P}\sum_{n=3} a_n\left(\frac{R}{R_0}\right)^n \frac{dP_n}{d\vartheta}(\pi/2) \tag{20a}$$

$$\gamma > \gamma_c \cong 2R_0 - \frac{q}{P}\sum \frac{a_n}{n}\left(\frac{R}{R_0}\right)^n \frac{dP_n}{d\vartheta}(\pi/2) \tag{20b}$$

The vertical cutoff momentum for orbits reaching the sphere $r = R$ at colatitude ϑ is then

$$P_c \cong \frac{qM \sin^4 \vartheta}{4R^2}\left[1 + \frac{2R}{M \sin \vartheta} \sum \frac{a_n}{n} \cdot \left\{\frac{\sin^{2n-1} \vartheta}{2^n} \frac{dP_n}{d\vartheta}(\pi/2) - \frac{dP_n}{d\vartheta}(\vartheta)\right\}\right] \qquad (21)$$

Comparison with the Quenby-Webber theory. A theory dealing with the motion of charged particles in a general perturbed dipole field has been developed by *Quenby and Webber* [1959]. Though we cannot use the preceding results to check this theory in the general, nonaxisymmetric case, we can do so for axisymmetric fields.

Instead of the harmonic expansion, Quenby and Webber used (close to the equatorial plane) functions $\Delta B_n(\vartheta)$ to describe the field; ΔB_n was defined as the horizontal component, at $r = R$, of that part of $\mathbf{B}$ which falls off as $r^{-(n+2)}$. The value of ΔB_n in an axisymmetric field is easily derived from the scalar potential and in the absence of external sources ($b_n = 0$) is

$$\Delta B_n(\vartheta) = -a_n R^{-2} \frac{dP_n}{d\vartheta} \qquad (22)$$

(In the actual paper, $\mathbf{H}$ and ΔH_n were used rather than $\mathbf{B}$ and ΔB_n.) With the notation of (19) and (22), (14) becomes

$$A = \frac{M \sin \vartheta}{r^2} + R \sum \left(\frac{R}{r}\right)^{n+1} \frac{\Delta B_n}{n} \qquad (23)$$

This agrees with (18) [*Quenby and Webber*, 1959], which gives the contribution of the quadrupole term ΔB_2 to A. Continuing by Treiman's method, the above authors calculated the effects of ΔB_2 on the critical γ and r and finally showed that the cutoff momentum in the equatorial plane becomes

$$P_c = \frac{qM}{4R^2}\left(1 + \frac{3}{4}\frac{R^3 \Delta B_2}{M}\right)$$

The analogous expressions for arbitrary n are not explicitly given, but it is stated that a similar modification of P_c is obtained (in the equatorial plane) with the difference that the factor 3/4 is replaced by

$$\frac{2}{n}(1 - 2^{-n})$$

This agrees with (21).

Unfortunately, in a later paper [*Webber*, 1963] in which the effects of higher-order terms are explicitly given, they are incorrectly transcribed (though they yield correct results for $n = 2$). They should be adjusted as follows (numbers refer to equations in the paper by *Webber* [1963]): In the last term of (9), the factor $\cos \lambda$ should be deleted, the last term of (10) should read

$$\left(1 + \frac{1}{\sin \vartheta} \sum \frac{\Delta H_n}{nH_0}\left(\frac{R_e}{R}\right)^{n-1}\right)$$

and that of (11)

$$\sum \frac{\Delta H_n}{nH_0} R_e^{n-1}$$

After appropriate changes in notation, these revised equations agree with the results given here.

Adiabatic invariance. Let an axisymmetric magnetic field satisfying (11*c*) and (12) (e.g., a curl-free field) be given. Lagrange's equations and therefore (16) still hold when the magnetic field is time dependent, even though energy is no longer conserved on account of the induced electric field. The same argument leading to the neglect of $\cos \omega$ in (17) for low momentums then shows that, for low momentums, the second term of (16) is much larger than the first. Neglecting the first term completely gives

$$p_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} \cong qAr \sin \vartheta \cong \text{const} \qquad (24)$$

Equation 24 shows, in a time-dependent axisymmetric field, how low-energy particles shift from one magnetic shell to another; the line of force parameter α of (13) is then conserved. This result can be generalized as follows.

Suppose the axisymmetric field undergoes a perturbation which is now not only time dependent but also asymmetrical. Equation 16 and its low-energy limit (24) then no longer hold. However, since the motion of trapped particles in the unperturbed field can be regarded as periodic in the coordinate φ, the action integral

$$J_\varphi = \int_0^{2\pi} p_\varphi \, d\varphi$$

is adiabatically conserved [compare *Landau and Lifshitz*, 1951, p. 54, and 1960, section 49–50], where p_φ is a component of the canonical momentum and can be approximated for low momentums by (24). The element of arc length is

$$\mathbf{dl} = \mathbf{i}_r \, dr + \mathbf{i}_\vartheta r \, d\vartheta + \mathbf{i}_\varphi r \sin \vartheta \, d\varphi$$

so that

$$J_\varphi \cong q \oint \mathbf{A}\, \mathbf{dl}$$

with the integral extending over one rotation of φ. By Stokes' theorem

$$J_\varphi \cong q \int \text{curl } \mathbf{A}\, \mathbf{ds} = q\Phi$$

where Φ denotes the flux enclosed by the shell to which the particle is attached. Thus we obtain the flux invariant, or third adiabatic invariant [*Northrop and Teller*, 1960], as a generalization of Störmer's integral; in a perturbed axisymmetric field, the magnetic flux through a magnetic shell is adiabatically conserved.

Appendix

Treiman's approach is the following. By (17), for any given γ and v the accessible region in the (r, ϑ) plane is bounded by lines where cos ω equals 1 or (-1). Of these (in fields not deviating much from a dipole) the former

$$\frac{\gamma}{r \sin \vartheta} - \frac{qA}{P} = 1 \tag{25}$$

determine whether trapping occurs. Regarding γ as a parameter, *Treiman* [1953] showed that trapping just starts when, in the equatorial plane, (25) acquires a double root for r. In near-dipole fields without external sources, this occurs when, for $\sin \vartheta = 1$,

$$\partial\gamma/\partial r = 0$$

(when external sources exist this may not hold [*Ray*, 1956]). For purposes of calculation, it is useful here to split A into two parts, A_1 giving the dipole field and A_2 (small by comparison) the higher terms. If M is defined as in (19)

$$A_1 = M \sin \vartheta / r^2 \tag{26}$$

Putting $\vartheta = \pi/2$ and neglecting all externa[l] sources, (25) becomes

$$\gamma = r + \frac{qM}{P}\frac{1}{r} - \frac{q}{P}\sum_{n=2} \frac{a_n}{n}\left(\frac{R}{r}\right)^n \frac{dP_n}{d\vartheta}(\pi/2) \tag{27}$$

Since $dP_n/d\vartheta$ vanishes in the equatorial plane for any even n, only odd values of n need to be considered in the last term. Let (25) be satisfied at $r = R_1$; then

$$\frac{qM}{P}\frac{1}{R_1^2} - \frac{q}{PR}\sum_{n=3} a_n\left(\frac{R}{R_1}\right)^{n+1} \frac{dP_n}{d\vartheta}(\pi/2) = 1 \tag{28}$$

As a first approximation, let the higher terms be neglected. Then

$$R_1 \cong R_0 = (qM/P)^{1/2} \tag{29}$$

R_0 is the well-known Störmer unit of length. Now let

$$R_1 = R_0(1 + \delta) \tag{30}$$

Collecting all first-order terms in (25) gives

$$\delta \cong -\frac{q}{2PR}\sum_{n=3} a_n\left(\frac{R}{R_0}\right)^{n+1} \frac{dP_n}{d\vartheta}(\pi/2) \tag{31}$$

However, substituting (30) in (27) shows that to first order of approximation the critical γ (denoted γ_c) does not depend on δ.

$$\gamma_c = 2R_0 - \frac{q}{P}\sum_{n=3} \frac{a_n}{n}\left(\frac{R}{R_0}\right)^n \frac{dP_n}{d\vartheta}(\pi/2) \tag{32}$$

It does, however, depend on the momentum P (assume for simplicity all particles are identical, e.g. protons) both directly and through R_0, and represents the limit of complete trapping for this momentum. Suppose now that such marginally trapped particles hit the earth $(r = R)$ vertically $(\cos \omega = 0)$ at colatitude ϑ; they then represent the vertical cutoff momentum P_c at that colatitude and by (17)

$$\gamma_c = \frac{qr \sin \vartheta}{P}\left[\frac{M \sin \vartheta}{R^2} - \frac{1}{R}\sum_{n=2} \frac{a_n}{n}\frac{dP_n}{d\vartheta}\right] \tag{33}$$

Neglecting nondipole components, we obtain from (32) and (33) as a first approximation

$$R/R_0 = \tfrac{1}{2}\sin^2 \vartheta \tag{34}$$

This approximation is inserted into the correction terms of (32) and (33), giving

$$P_c \cong Mq\left[\frac{\sin^2 \vartheta}{2R} + \frac{\sin \vartheta}{2M}\sum_{n=2} \frac{a_n}{n} \cdot\left\{\frac{\sin^{2n-1} \vartheta}{2^n}\frac{dP_n}{d\vartheta}(\pi/2) - \frac{dP_n}{d\vartheta}(\vartheta)\right\}\right]^2 \tag{35}$$

This differs from (21) only by second-order terms.

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